properties of Continuous time Fourier Transform.

(1) Linearity property:

The linearity properties states that the weighted sum of two signal is equal to weighted sum of their individual transform.

i.e If

$$x_1(t) \xleftrightarrow{FT} X_1(\omega)$$

and $$x_2(t) \longleftrightarrow X_2(\omega)$$

Then,

$$\boxed{a_1 x_1(t) + b x_2(t) \xleftrightarrow{FT} a_1 X_1(\omega) + b X_2(\omega).}$$

where a and b are constants.

proof:

$$F[a_1 x_1(t) + b x_2(t)] = \int_{-\infty}^{+\infty} [a x_1(t) + b x_2(t)] \cdot e^{-j\omega t} dt$$

$$= \int_{-\infty}^{+\infty} a\, x_1(t) e^{-j\omega t} dt + b \int_{-\infty}^{\infty} x_2(t) \cdot e^{-j\omega t} dt$$

$$= a X_1(\omega) + b X_2(\omega).$$

Proved

(2) Time shifting property:

The time shifting property states that if a signal $x(t)$ is shifted by t_0 sec, the spectrum is modified by a linear phase shift of slope $-\omega t_0$

i.e If $x(t) \xleftrightarrow{FT} X(\omega)$

Then, $\boxed{x(t-t_0) \xleftrightarrow{FT} e^{-j\omega t_0} X(\omega).}$

proof:

$\therefore F[x(t)] = \int_{-\infty}^{+\infty} x(t) \cdot e^{-j\omega t} dt$

now, delay the signal by t_0 times. $x(t) \xrightarrow{\text{delay } t_0} x(t-t_0)$.

$$F[x(t-t_0)] = \int_{-\infty}^{+\infty} x(t-t_0) \cdot e^{-j\omega t} dt$$

put $t - t_0 = p$

$t = t_0 + p$

$$F[x(p)] = \int_{-\infty}^{+\infty} x(p) \cdot e^{-j\omega(t_0+p)}$$

$$= \int_{-\infty}^{+\infty} \underline{x(p)} \cdot e^{-j\omega t_0} \cdot \underline{e^{-j\omega p}} \cdot dp, \quad \int_{-\infty}^{+\infty} x(p) e^{-j\omega p} dp$$

$$F[x(t-t_0)] = X(\omega) \cdot e^{-j\omega t_0}$$

$\therefore x(t-t_0) \xleftrightarrow{FT} e^{-j\omega t_0} \cdot X(\omega)$.

Similarly, $\boxed{x(t+t_0) \xleftrightarrow{FT} e^{j\omega t_0} \cdot X(\omega).}$ Proved

Here There is no change in magnitude spectrum but the phase spectrum is linearly shifted.

(3) Frequency shifting property (Multiplication by an exponential)

Frequency shifting property states that the multiplication of a time domain signal $x(t)$ by $e^{j\omega_0 t}$ results in frequency spectrum shifted by ω_0 i.e

If

$x(t) \xleftrightarrow{FT} X(\omega)$.

Then, $\boxed{e^{j\omega_0 t} x(t) \longleftrightarrow X(\omega - \omega_0).}$

proof:

$\because \quad F[x(t)] = \int_{-\infty}^{\infty} x(t)\, e^{-j\omega t}\, dt$

then, $F[e^{j\omega_0 t} x(t)] = \int_{-\infty}^{+\infty} e^{j\omega_0 t} x(t)\, e^{-j\omega t}\, dt$

$= \int_{-\infty}^{+\infty} e^{j(\omega_0 - \omega)t}\, dt$

$= \int_{-\infty}^{+\infty} e^{-j(\omega - \omega_0)t}\, dt$

$= X(\omega - \omega_0)$

$\therefore \quad \boxed{e^{j\omega_0 t} x(t) \xleftrightarrow{FT} X(\omega - \omega_0).}$

$e^{-j\omega t_0}$ Similarly $\boxed{e^{-j\omega_0 t} x(t) \longleftrightarrow X(\omega + \omega_0)}$ proved

(4) Time reversal property:

The time reversal property states that.

If $x(t) \xleftrightarrow{FT} X(\omega)$.

Then $\boxed{x(-t) \xleftrightarrow{FT} X(-\omega).}$

proof:

$\because \quad F[x(t)] = \int_{-\infty}^{+\infty} x(t) \cdot e^{-j\omega t}\, dt$.

then

$F[x(-t)] = \int_{-\infty}^{+\infty} x(-t) \cdot e^{-j\omega t}\, dt$

$= X(-\omega)$. proved

(5) Time scaling property:

If $x(t) \xleftrightarrow{FT} X(\omega)$.

then $\boxed{x(at) \xleftrightarrow{FT} \frac{1}{|a|} X\left(\frac{\omega}{a}\right).}$

proof: ∴ we know that

$$F[x(t)] = \int_{-\infty}^{+\infty} x(t) \cdot e^{-j\omega t} dt$$

Then, ~~for~~

$$F[x(at)] = \int_{-\infty}^{+\infty} x(at)\, e^{-j\omega at} dt$$

put $at = p$.
on differentiating w.r.to t we get,

$$a = \frac{dp}{dt}$$

$$\therefore dt = \frac{dp}{a}$$

$$\therefore F[x(at)] = \int_{-\infty}^{+\infty} \frac{x(p)}{a} \cdot e^{-j\frac{\omega p}{a}} dp$$

$$= \frac{1}{a} \int_{-\infty}^{+\infty} x(p) \cdot e^{-j\frac{\omega p}{a}} dp.$$

Case-I when $a > 0$.

Then $$F[x(at)] = \frac{1}{a} \int_{-\infty}^{+\infty} x(p) \cdot e^{-j\left(\frac{\omega}{a}\right)p} dp.$$

$$= \frac{1}{a} X\left(\frac{\omega}{a}\right).$$

proved

Case-II when $a < 0$, then

$$F[x(at)] = \frac{1}{-a} \int_{-\infty}^{+\infty} x(p) \cdot e^{-j\left(\frac{\omega}{a}\right)p} dp.$$

$$= \frac{-1}{a} X\left(\frac{\omega}{a}\right).$$

$$= \frac{1}{|a|} X\left(\frac{\omega}{a}\right)$$

$$\therefore x(at) \xleftrightarrow{FT} \frac{1}{|a|} X\left(\frac{\omega}{a}\right)$$ proved

6. Differentiation in Time Domain property.

The differentiation in time domain property states that the differentiation of a function in time domain is equivalent to the ~~multip~~ product of $X(\omega)$ by $j\omega$.

i.e,

If $x(t) \xleftrightarrow{FT} X(\omega)$.

then,

$$\boxed{\frac{d}{dt}x(t) \xleftrightarrow{FT} j\omega X(\omega).}$$

proof: By definition,

$$\therefore\ x(t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} X(\omega)\, e^{j\omega t} d\omega.$$

Differentiating w.r.to t we get:

$$\frac{d\, x(t)}{dt} = \frac{1}{2\pi}\frac{d}{dt}\left[\int_{-\infty}^{+\infty} X(\omega)\cdot e^{j\omega t} d\omega\right]$$

$$= \frac{1}{2\pi}\int_{-\infty}^{+\infty} X(\omega)\cdot \frac{d}{dt}[e^{j\omega t}]\, d\omega.$$

$$= \frac{1}{2\pi}\int_{-\infty}^{+\infty} X(\omega)\cdot e^{j\omega t}\cdot j\omega\cdot d\omega.$$

$$= j\omega\left[\frac{1}{2\pi}\int_{-\infty}^{+\infty} X(\omega)\, e^{j\omega t} d\omega\right]$$

$$= j\omega\cdot F^{-1}[X(\omega)]$$

$$= \cancel{j\omega\, x(t)}\ j\omega\, X(\omega).$$

$$\therefore\ \frac{d\, x(t)}{dt} = j\omega\, X(\omega).$$ proved

In general $$\frac{d^n x(t)}{dt^n} \xleftrightarrow{FT} (j\omega)^n X(\omega).$$

7. Differentiation in frequency domain property:

The differentiation in frequency domain property states that the multiplication of a signal $x(t)$ by t is equivalent to differentiation of its FT in frequency domain i.e

If $\quad x(t) \xleftrightarrow{FT} X(\omega).$

Then $\quad \boxed{t\,x(t) \xleftrightarrow{FT} j\frac{d}{d\omega} X(\omega).}$

proofs: By definition

$$X(\omega) = \int_{-\infty}^{+\infty} x(t)\, e^{-j\omega t}\, dt$$

Now, differentiating on the both sides we get:

$$\frac{d}{d\omega}X(\omega) = \frac{d}{d\omega}\left[\int_{-\infty}^{+\infty} x(t).\,e^{-j\omega t}\, dt\right]$$

$$= x(t)\frac{d}{d\omega}\int_{-\infty}^{+\infty} e^{-j\omega t}\, dt$$

$$= x(t)\cdot\int_{-\infty}^{+\infty} e^{-j\omega t}\cdot -jt\, dt$$

$$= \cancel{j\omega \int_{-\infty}^{+\infty} x(t).\,e^{-j\omega t}\, dt} = x(t)\cdot\int_{-\infty}^{+\infty} e^{-j\omega t}\cdot -jt\, dt$$

$$j\frac{d X(\omega)}{d(\omega)} = \cancel{j\omega X(\omega)} \times j = -x(t)\cdot jt\int_{-\infty} e^{-j\omega t}\cdot dt$$

$$= \cancel{\omega X(\omega)} = -jt\int_{-\infty}^{+\infty} x(t)\cdot e^{-j\omega t}\, dt$$

$$j\frac{d}{d\omega}X(\omega) = -jt\int_{-\infty}^{+\infty} x(t)\cdot e^{-j\omega t}\, dt\,(\times j)$$

$$= \int_{-\infty}^{+\infty} t\,x(t)\cdot e^{-j\omega t}\, dt$$

$$j\frac{d}{d\omega}X(\omega) = FT[t\,x(t)]$$

$$\therefore \quad t\,x(t) \xleftrightarrow{FT} j\frac{d}{d\omega}X(\omega)$$

proved

8. Time Integration property.

Time integration property: The time integration property states that the integration of a function $x(t)$ in time is equivalent to the division of its FT by $j\omega$. i.e

If

$$x(t) \xleftrightarrow{FT} X(\omega).$$

Then,

$$\boxed{\int_{-\infty}^{t} x(\tau)\cdot d\tau \xleftrightarrow{FT} \frac{1}{j\omega} X(j\omega) \quad \text{if } X(0) = 0.}$$

~~9. Time Integration property.~~

~~The time integration property states that the integration of a function $x(t)$ in time domain is equivalent to the ~~differentiation of~~ division of its Fourier transform by $j\omega$ i.e~~

~~If $x(t) \xleftrightarrow{FT} X(\omega)$.~~

~~Then, $\int_{-\infty}^{t} x(\tau)\, d\tau \xleftrightarrow{FT} \frac{1}{j\omega} X(\omega)$ if $X(0) = 0$.~~

Proof:

$\because$ $$x(t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} X(\omega)\cdot e^{j\omega t}\, d\omega.$$

Now, replace t by a dummy variable τ

then,

$$x(\tau) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} X(\omega)\cdot e^{j\omega\tau}\, d\omega$$

Now, Integrating on the both sides w.r.t τ and take the limit of integration from $-\infty$ to t

$$\int_{-\infty}^{t} x(\tau)\cdot d\tau = \int_{-\infty}^{t}\left[\frac{1}{2\pi}\int_{-\infty}^{+\infty} X(\omega)\cdot e^{j\omega\tau}\, d\omega\right] d\tau$$

Now, Interchang the order of integration, we get.

$$\int_{-\infty}^{t} x(\tau)\cdot d\tau = \frac{1}{2\pi}\int_{-\infty}^{+\infty} X(\omega)\cdot\left(\int_{-\infty}^{t} e^{j\omega\tau}\cdot d\tau\right)d\omega.$$

$$= \frac{1}{2\pi}\int_{-\infty}^{+\infty} X(\omega)\cdot\left[\frac{e^{j\omega\tau}}{j\omega}\right]_{-\infty}^{t}\cdot d\omega$$

$$= \frac{1}{2\pi}\int_{-\infty}^{+\infty} X(\omega)\cdot\frac{e^{j\omega t}}{j\omega}\cdot d\omega.$$

$$\frac{1}{2\pi j\omega} * \int_{-\infty}^{+\infty} X(\omega)\cdot e^{j\omega t}\cdot d\omega.$$

$$\frac{1}{j\omega}\cdot\frac{1}{2\pi}\int_{-\infty}^{+\infty} X(\omega)\, e^{j\omega t}\, d\omega$$

$$\int_{-\infty}^{t} x(\tau)\cdot d\tau = \cancel{\frac{1}{j\omega}\, F^{-1}\left[X(\omega)\right]}\; F^{-1}\left[\frac{X(\omega)}{j\omega}\right]$$

or,

$$F\left[\int_{-\infty}^{t} x(\tau)\cdot d\tau\right] = \frac{X(\omega)}{j\omega}.$$ proved

(9.) Convolution property or Theorem

The convolution property or theorem states that the convolution of two signals in time domain is equivalent to the multiplication of their spectra in frequency domain. This is also known as the time convolution theorem.

i.e

If $\quad x_1(t) \xleftrightarrow{FT} X_1(\omega).$

and $\quad x_2(t) \xleftrightarrow{FT} X_2(\omega).$

Then,

$$\boxed{x_1(t) * x_2(t) \xleftrightarrow{FT} X_1(\omega)\cdot X_2(\omega).}$$

Proof: (i) as we know that the convolution of two signals $x_1(t)$ and $x_2(t)$ is given by

$$x_1(t) * x_2(t) = \int^{+\infty} x_1(\tau)\, x_2(t-\tau)\cdot d\tau.$$

Now, taking the Fourier Transform on the both sides we get:

$$F[x_1(t)\cdot x_2(t)] = \int_{-\infty}^{+\infty} \cancel{x(\tau)\cdot x_2(t)}\, x_1(t)\cdot x_2(t-\tau).$$

$$\int_{-\infty}^{+\infty} x_1(t)\cdot x_2(t)\cdot e^{-j\omega t}\, dt$$

$$x_1(t) * x_2(t) = \int_{-\infty}^{+\infty} x_1(\tau)\cdot x_2(t-\tau)\cdot d\tau.$$

~~Now~~ Now, taking the Fourier Transform we get:

$$F[x_1(t) * x_2(t)] = \int_{-\infty}^{+\infty}\left[\int_{-\infty}^{+\infty} x_1(\tau)\cdot x_2(t-\tau)\cdot d\tau\right]\cdot e^{-j\omega t}\cdot dt$$

Now, on ~~integrating~~ interchanging the order of integration, we get.

$$F[x_1(t)\cdot x_2(t)] = \int_{-\infty}^{+\infty} x_1(\tau)\cdot\left[\int_{-\infty}^{+\infty} x_2(t-\tau)\cdot e^{-j\omega t}\, dt\right] d\tau$$

Now put $t-\tau = p$ in the second integration and diff. w.r.t. t we

$$\cancel{F}\quad \cancel{\frac{dt}{dp} - 0 = dp} \qquad 1-0 = \frac{dp}{dt}$$

$$dt = dp$$

$$F[x_1(t)\cdot x_2(t)] = \int_{-\infty}^{+\infty} x_1(\tau)\cdot\left[\int_{-\infty}^{+\infty} x_2(p)\cdot e^{-j\omega(p+\tau)}\, dp\right] d\tau.$$

$$= \int_{-\infty}^{+\infty} x_1(\tau)\left[\int_{-\infty}^{+\infty} x_2(p)\cdot e^{-j\omega p}\, dp\right]\cdot e^{-j\omega\tau}\, d\tau$$

$$= \int_{-\infty}^{+\infty} x_1(\tau)\cdot e^{-j\omega\tau}\, d\tau\cdot X_2(\omega).$$

$$= X_2(\omega)\cdot\int_{-\infty}^{+\infty} x_1(\tau)\cdot e^{-j\omega\tau}\, d\tau$$

$$= X_2(\omega)\cdot X_1(\omega).$$

$$\boxed{x_1(t) * x_2(t) \xleftrightarrow{FT} X_1(\omega)\, X_2(\omega)}$$

proved

(10) Multiplication property or Theorem.

This property states that the multiplication of two function in time domain is equivalent to the convolution of their spectra in the frequency domain. This is also called the frequency convolution theorem.

If $x_1(t) \xleftrightarrow{FT} X_1(\omega)$ and

$x_2(t) \xleftrightarrow{FT} X_2(\omega)$.

Then, $$\boxed{x_1(t) \cdot x_2(t) \xleftrightarrow{FT} \frac{1}{2\pi} X_1(\omega) * X_2(\omega)}$$

proof:

as we know that

$$F[x(t)] = X(\omega) = \int_{-\infty}^{+\infty} x(t) \cdot e^{-j\omega t}\, dt$$

and $x(t) = \int_{-\infty}^{+\infty} X(\omega)\, e^{j\omega t}\, d\omega$

Then,

$$F[x_1(t) \cdot x_2(t)] = \int_{-\infty}^{+\infty} x_1(t) \cdot x_2(t)\, e^{-j\omega t}\, dt$$

Now, on putting the $x_1(t)$ ~~in term~~

$= x_1(t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} X(\lambda) \cdot e^{j\lambda t}\, d\lambda$ in above equation.

$$F[x_1(t) \cdot x_2(t)] = \int_{-\infty}^{+\infty} \left[\frac{1}{2\pi} \int_{-\infty}^{+\infty} X(\lambda) \cdot e^{j\lambda t}\, d\lambda \right] x_2(t) \cdot e^{-j\omega t}\, dt$$

Now, on interchanging the orders of integration we get.

$$F[x_1(t) \cdot x_2(t)] = \frac{1}{2\pi} \int_{-\infty}^{+\infty} X_1(\lambda) \cdot \left[\int_{-\infty}^{+\infty} x_2(t) \cdot e^{-j\omega t} \cdot e^{j\lambda t}\, dt \right]$$

$$= \frac{1}{2\pi} \int_{-\infty}^{+\infty} X_1(\lambda) \cdot \left[\int_{-\infty}^{+\infty} x_2(t) \cdot e^{j(\lambda - \omega) t}\, dt \right]$$

$$= \frac{1}{2\pi} \int_{-\infty}^{+\infty} X_1(\lambda) \left[\int_{-\infty}^{+\infty} x_2(t) \cdot e^{-(\omega - \lambda) t}\, dt \right] d\lambda$$

$$= \frac{1}{2\pi}\int_{-\infty}^{+\infty} X_1(\lambda) \cdot X_2(\omega-\lambda)\cdot d\lambda$$

$$= \frac{1}{2\pi} X_1(\omega) * X_2(\omega)$$

$$\therefore \quad F[x_1(t)\cdot x_2(t)] = \frac{1}{2\pi} X_1(\omega) * X_2(\omega)$$

$$= X_1(f) * X_2(f)$$ Proved.